1. We consider a Markov chain defined on $\{x_1, x_2, x_3\}$ and the transition matrix is

$$T = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/5 & 1/5 & 3/5 \\ 1/2 & 1/4 & 1/4 \end{pmatrix}.$$

- (a) Suppose that the initial probability distribution is $\pi^1 = (0.2,\, 0.2,\, 0.6)$, calculate the marginal distributions at $t = 2, 2, 3, 4, 5$. You may use a calculator and keep two decimal places.
- (b) Suppose that the initial probability distribution is $\pi^1 = (0.4,\, 0.1,\, 0.5)$, calculate the marginal distributions at $t = 2, 2, 3, 4, 5$. You may use a calculator and keep two decimal places.
- (c) Compute the stationary distribution of the Markov chain.

Solution: (a)

$$\pi^1 = (0.2,\, 0.2,\, 0.6)$$

$$\pi^2 = \pi^1 T = (0.2,\, 0.2,\, 0.6) \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/5 & 1/5 & 3/5 \\ 1/2 & 1/4 & 1/4 \end{pmatrix} = [0.41,\, 0.26,\, 0.34]$$

$$\pi^3 = \pi^2 T = [0.36,\, 0.27,\, 0.37]$$

$$\pi^4 = \pi^3 T = [0.36,\, 0.27,\, 0.37]$$

$$\pi^5 = \pi^4 T = [0.36,\, 0.27,\, 0.37]$$

(b): The procedure is the same as (a) and the results are:

t	Marginal Distribution (Case b)
2	$(0.40, 0.28, 0.32)$
3	$(0.35, 0.27, 0.38)$
4	$(0.36, 0.27, 0.37)$
5	$(0.36, 0.27, 0.37)$

(c) To compute the stationary distribution of a Markov chain, we solve for the distribution vector $\pi = (\pi_1, \pi_2, \pi_3)$ such that:

$$\pi T = \pi,$$

where T is the given transition matrix:

$$T = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{5} & \frac{1}{5} & \frac{3}{5} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

This implies solving the system of linear equations:

$$\begin{aligned}
\pi_1 \cdot \frac{1}{3} + \pi_2 \cdot \frac{1}{5} + \pi_3 \cdot \frac{1}{2} &= \pi_1, \\
\pi_1 \cdot \frac{1}{3} + \pi_2 \cdot \frac{1}{5} + \pi_3 \cdot \frac{1}{4} &= \pi_2, \\
\pi_1 \cdot \frac{1}{3} + \pi_2 \cdot \frac{3}{5} + \pi_3 \cdot \frac{1}{4} &= \pi_3.
\end{aligned}$$

After simplification it becomes:

$$\begin{aligned}
-\frac{2}{3}\pi_1 + \frac{1}{5}\pi_2 + \frac{1}{2}\pi_3 &= 0, \\
\frac{1}{3}\pi_1 - \frac{4}{5}\pi_2 + \frac{1}{4}\pi_3 &= 0, \\
\frac{1}{3}\pi_1 + \frac{3}{5}\pi_2 - \frac{3}{4}\pi_3 &= 0.
\end{aligned}$$

For a linear system like this (Ax=0), we know that if the coefficient matrix A is full rank, it has a unique solution $x = 0$, which is obviously not what we want. If the system is not full rank, it has infinitely many solutions. In fact, one can compute the rank of the matrix which is 2.

Therefore we will take two equations out of the three – in principle you can take any two, and we usually choose the two that we think are simple. In this problem we just take the first two. Additionally, we know the probabilities must sum to 1:

$$\pi_1 + \pi_2 + \pi_3 = 1.$$

We put the three equations together, yielding the following linear system:

$$\begin{aligned}
-\frac{2}{3}\pi_1 + \frac{1}{5}\pi_2 + \frac{1}{2}\pi_3 &= 0, \\
\frac{1}{3}\pi_1 - \frac{4}{5}\pi_2 + \frac{1}{4}\pi_3 &= 0, \\
\pi_1 + \pi_2 + \pi_3 &= 1.
\end{aligned}$$

Solving this linear system we find

$$\pi = \begin{bmatrix} 0.36 \\ 0.27 \\ 0.37 \end{bmatrix}.$$