1. The probability density function (PDF) of a random variable X is given by:

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

- (a) Find the cumulative distribution function (CDF) $F(x)$.
- (b) Find the inverse CDF $F^{-1}(u)$ for $u \in [0, 1]$.
- (c) Using the inverse CDF, calculate the value of X if:
  - $u = 0.2$,
  - $u = 0.8$.
- (d) Verify that if $u \sim U[0, 1]$, $x = F^{-1}(u)$ follows the PDF $f(x)$.

Solution:

- **Find the CDF $F(x)$:**
  The CDF is the integral of the PDF:

$$F(x) = \int_0^x f(t)\,dt = \int_0^x 2t\,dt = \left[t^2\right]_0^x = x^2 \quad \text{for } 0 \leq x \leq 1.$$

  For all x, the CDF is:

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ x^2 & \text{if } 0 \leq x \leq 1, \\ 1 & \text{if } x > 1. \end{cases}$$

- **Find the inverse CDF $F^{-1}(u)$:**
  For $u \in [0, 1]$, solve $F(x) = u$:

$$u = x^2 \implies x = \sqrt{u}.$$

  Thus, the inverse CDF is:

$$F^{-1}(u) = \sqrt{u}.$$

- **Calculate X for given u:**
  - For $u = 0.2$:

$$X = \sqrt{0.2} \approx 0.447.$$

  - For $u = 0.8$:

$$X = \sqrt{0.8} \approx 0.894.$$

- **Derive PDF:**

  We aim to derive the PDF of $X = \sqrt{U}$, where $U \sim \text{Uniform}(0, 1)$.

  The PDF of U is:

$$f_U(u) = \begin{cases} 1, & \text{if } 0 \leq u \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

  The transformation is $X = \sqrt{U}$, or equivalently $U = X^2$. Using the transformation formula:

$$f_X(x) = f_U(u) \cdot \left| \frac{du}{dx} \right|,$$

  where $u = X^2$.

  From $u = X^2$, the derivative is:

$$\frac{du}{dx} = 2x.$$

  Substituting $f_U(u) = 1$ and $\frac{du}{dx} = 2x$, we get:

$$f_X(x) = 1 \cdot |2x| = 2x.$$

  The range of X is determined by the range of U. Since $U \in [0, 1]$ and $X = \sqrt{U}$, $X \in [0, 1]$. Thus:

$$f_X(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

  Finally, verify normalization:

$$\int_0^1 f_X(x)\, dx = \int_0^1 2x\, dx = \left[x^2\right]_0^1 = 1.$$

  The PDF of $X = \sqrt{U}$ is:

$$f_X(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

2. Show that if $\mathbf{x} \sim N(\mathbf{0}, \mathbf{I})$, then $\mathbf{z} = \mathbf{L}\mathbf{x} + \boldsymbol{\mu}$ (L invertible) is multivariate normal with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma} = \mathbf{L}\mathbf{L}^T$.

   Solution:

   Let:

$$\mathbf{z} = \mathbf{L}\mathbf{x} + \boldsymbol{\mu}.$$

Rewriting $\mathbf{x}$ in terms of $\mathbf{z}$:

$$\mathbf{x} = \mathbf{L}^{-1}(\mathbf{z} - \boldsymbol{\mu}),$$

as $\mathbf{L}$ is invertible.

The PDF of $\mathbf{x} \sim N(\mathbf{0}, \mathbf{I})$ is:

$$f_{\mathbf{x}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\mathbf{x}^T\mathbf{x}}.$$

Using the change of variables formula:

$$f_{\mathbf{z}}(\mathbf{z}) = f_{\mathbf{x}}(\mathbf{L}^{-1}(\mathbf{z} - \boldsymbol{\mu})) \cdot \left|\det(\mathbf{L}^{-1})\right|.$$

and

$$f_{\mathbf{z}}(\mathbf{z}) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}(\mathbf{L}^{-1}(\mathbf{z}-\boldsymbol{\mu}))^T(\mathbf{L}^{-1}(\mathbf{z}-\boldsymbol{\mu}))} \cdot \left|\det(\mathbf{L}^{-1})\right|.$$

Simplify:

$$f_{\mathbf{z}}(\mathbf{z}) = \frac{1}{(2\pi)^{n/2}|\det(\mathbf{L})|} e^{-\frac{1}{2}(\mathbf{z}-\boldsymbol{\mu})^T(\mathbf{L}^{-1})^T\mathbf{L}^{-1}(\mathbf{z}-\boldsymbol{\mu})}.$$

The mean of $\mathbf{z}$ is:

$$\mathbb{E}[\mathbf{z}] = \mathbb{E}[\mathbf{L}\mathbf{x} + \boldsymbol{\mu}] = \mathbf{L}\mathbb{E}[\mathbf{x}] + \boldsymbol{\mu}.$$

Since $\mathbb{E}[\mathbf{x}] = \mathbf{0}$:

$$\mathbb{E}[\mathbf{z}] = \boldsymbol{\mu}.$$

The covariance of $\mathbf{z}$ is:

$$\mathrm{Cov}(\mathbf{z}) = \mathbb{E}[(\mathbf{z} - \mathbb{E}[\mathbf{z}])(\mathbf{z} - \mathbb{E}[\mathbf{z}])^T].$$

Substituting $\mathbf{z} - \mathbb{E}[\mathbf{z}] = \mathbf{L}\mathbf{x}$:

$$\mathrm{Cov}(\mathbf{z}) = \mathbb{E}[\mathbf{L}\mathbf{x}(\mathbf{L}\mathbf{x})^T] = \mathbf{L}\mathbb{E}[\mathbf{x}\mathbf{x}^T]\mathbf{L}^T.$$

Since $\mathbb{E}[\mathbf{x}\mathbf{x}^T] = \mathbf{I}$ (identity matrix):

$$\mathrm{Cov}(\mathbf{z}) = \mathbf{L}\mathbf{I}\mathbf{L}^T = \mathbf{L}\mathbf{L}^T.$$