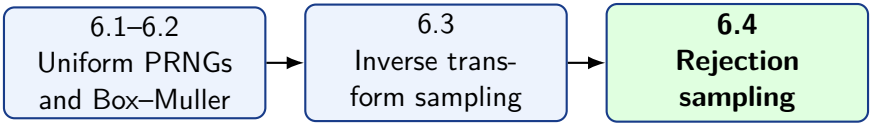
Computational Statistics

Sampling a Univariate Distribution

Section 6.4: Rejection Sampling

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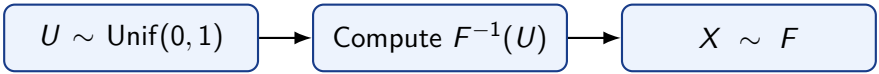
Common objective

Generate samples from a target density $p(x)$ using random numbers from distributions that are easy to sample.

Recap: inverse transform sampling (Section 6.3)

For a target cumulative distribution function F:

$$U \sim \text{Unif}(0,1), \qquad X = F^{-1}(U) \quad \Longrightarrow \quad X \sim F.$$



Potential obstacle

The CDF F or quantile function F^{-1} may be difficult to compute.

Why do we need another sampling method?

- For $X \sim \text{Exp}(\lambda)$, the quantile is simple:

$$F^{-1}(u) = -\frac{\log(1-u)}{\lambda}.$$

- For normal, beta and gamma distributions, the quantile generally has no elementary closed form.
- Numerical inversion is possible, but repeatedly solving $F(x) = u$ may be costly.

New idea

Instead of inverting F, generate an **easy proposal** and decide whether to keep it.

No closed-form inverse does *not* mean sampling is impossible; it motivates alternatives.

Rejection sampling: target, proposal and envelope

Our target density is $p(x)$, which is difficult to sample directly.

- Choose a **proposal density** $q(x)$ from which sampling is easy.
- Choose an **envelope constant** $c > 0$ such that

$$\boxed{p(x) \leq cq(x) \qquad \text{for all } x.}$$

- Equivalently, where $q(x) > 0$,

$$\boxed{c \geq \sup_{x:q(x)>0} \frac{p(x)}{q(x)}.}$$

Support requirement

Whenever $p(x) > 0$, we must also have $q(x) > 0$; otherwise no finite envelope can work.

The acceptance probability

Given a proposed value $Y = y$ from q, define

$$\alpha(y) = \frac{p(y)}{cq(y)}.$$

- The envelope condition guarantees $0 \leq \alpha(y) \leq 1$.
- Draw a fresh $U \sim \text{Unif}(0, 1)$ independently of Y.
- **Accept** the proposal if $U \leq \alpha(Y)$; otherwise **reject** and try again.

Important distinction

Y is a candidate from q; the *accepted* value X follows the target density p.

Algorithm 6.4: rejection sampling

Inputs

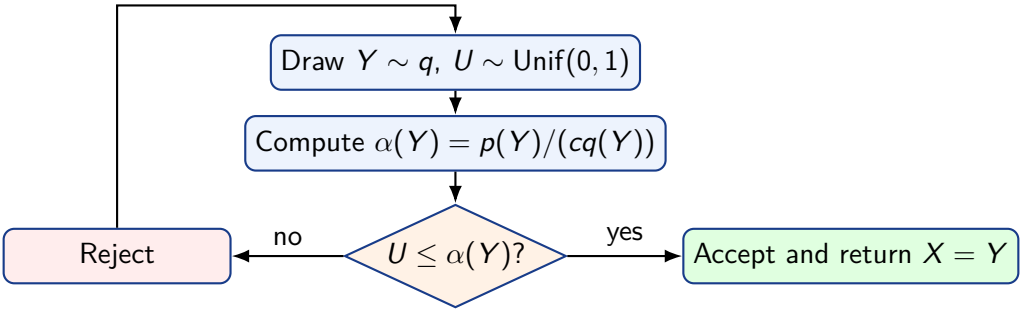
Target p, proposal q, and constant c satisfying $p(x) \leq cq(x)$.

1. Generate $Y \sim q$.
2. Independently generate $U \sim \text{Unif}(0, 1)$.
3. Calculate $\alpha(Y) = \dfrac{p(Y)}{cq(Y)}$.
4. If $U \leq \alpha(Y)$, **accept** and output $X = Y$.
   Otherwise **reject** and return to Step 1.

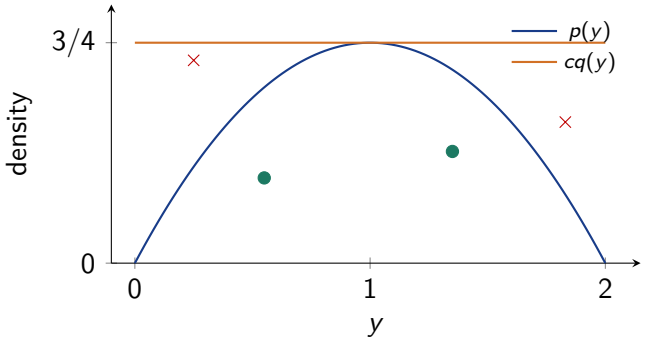
Guarantee

With a valid envelope, each accepted output has density p.

Algorithm as a flowchart



Geometric intuition: sample below an envelope



- The curve cq sits *above* p.
- Draw $Y \sim q$ and a uniform vertical height between 0 and $cq(Y)$.
- Accept only if the point lies under $p(Y)$.
- Green dots: accepted. Red crosses: rejected.

Diagram uses the quadratic example developed later; for a general q, the envelope need not be flat.

Let $Y \sim q$ and $U \sim \text{Unif}(0,1)$ be independent. Define

$$A = \left\{ U \leq \frac{p(Y)}{cq(Y)} \right\}$$

as the event that the proposal is accepted.

Claim

The distribution of the accepted value X is

$$\boxed{\mathbb{P}(X < x^*) = \int_{-\infty}^{x^*} p(t)\, dt.}$$

The proof in the lecture notes uses **conditional probability** and splits into a numerator and denominator.

Equation (6.5): condition on acceptance

Accepted values are proposals Y *conditional* on A:

$$\begin{aligned} \mathbb{P}(X < x^*) &= \mathbb{P}(Y < x^* \mid A) \\ &= \frac{\mathbb{P}(Y < x^*,\ A)}{\mathbb{P}(A)}. \end{aligned} \tag{6.5}$$

Numerator

$\mathbb{P}(Y < x^*,\ A)$

Proposal lies below x^* *and* is accepted.

Denominator

$\mathbb{P}(A)$

Overall probability that a proposal is accepted.

Equation (6.6): calculate the numerator

Condition on the candidate proposal $Y = t$:

$$\begin{aligned}
\mathbb{P}(Y < x^*,\ A) &= \int_{-\infty}^{x^*} \mathbb{P}(A \mid Y = t) q(t)\, dt \\
&= \int_{-\infty}^{x^*} \mathbb{P}\left(U \leq \frac{p(t)}{cq(t)}\right) q(t)\, dt \\
&= \int_{-\infty}^{x^*} \frac{p(t)}{cq(t)} q(t)\, dt \\
&= \boxed{\frac{1}{c} \int_{-\infty}^{x^*} p(t)\, dt.}
\end{aligned} \tag{6.6}$$

The penultimate equality uses $U \sim \text{Unif}(0, 1)$ and the valid bound $0 \leq p(t)/(cq(t)) \leq 1$.

Equation (6.7): calculate the denominator

Integrate over all possible proposal values:

$$\begin{aligned}\mathbb{P}(A) &= \int_{-\infty}^{\infty} \mathbb{P}(A \mid Y = t) q(t)\, dt \\ &= \int_{-\infty}^{\infty} \frac{p(t)}{cq(t)} q(t)\, dt \\ &= \frac{1}{c} \int_{-\infty}^{\infty} p(t)\, dt \\ &= \boxed{\frac{1}{c}.}\end{aligned} \tag{6.7}$$

Why does the final integral equal one?

p is a normalised probability density, so $\int p(t)\, dt = 1$.

Finish the proof: accepted samples have density p

Substituting (6.6) and (6.7) into (6.5),

$$\mathbb{P}(X < x^*) = \frac{\frac{1}{c}\int_{-\infty}^{x^*} p(t)\,dt}{\frac{1}{c}}$$

$$= \boxed{\int_{-\infty}^{x^*} p(t)\,dt.}$$

Conclusion

The CDF of the accepted value is the CDF of the target p. Therefore, X **has density** p.

The same argument works with $\leq$ rather than $<$; the distinction is immaterial for a continuous target.

Alternative one-line density argument

- A candidate at y has density $q(y)$.
- Given $Y = y$, it is accepted with probability $\alpha(y) = p(y)/(cq(y))$.

$$\underbrace{q(y)}_{\text{proposal density}} \times \underbrace{\frac{p(y)}{cq(y)}}_{\text{acceptance probability}} = \frac{p(y)}{c}.$$

Normalise by the overall acceptance probability $1/c$:

$$\boxed{p_{Y|A}(y) = \frac{p(y)/c}{1/c} = p(y).}$$

Acceptance probability: an exact formula

Because $Y \sim q$, and p, q are normalised densities,

$$\begin{aligned}\mathbb{P}(\text{accept}) &= \mathbb{E}_q\left[\frac{p(Y)}{cq(Y)}\right] \\ &= \int \frac{p(y)}{cq(y)} q(y)\, dy \\ &= \boxed{\frac{1}{c}.}\end{aligned}$$

Interpretation

$c = 1.5 \Rightarrow$ about 66.7% of proposals are accepted.
$c = 3 \Rightarrow$ about 33.3% are accepted.

Expected number of proposals per sample

For independent repeated proposals, each trial succeeds with probability

$$s = \frac{1}{c}.$$

The number of proposals T until the first acceptance follows a geometric distribution:

$$\mathbb{P}(T = k) = (1 - s)^{k-1} s, \qquad k = 1, 2, \ldots$$

Hence

$$\boxed{\mathbb{E}[T] = \frac{1}{s} = c.}$$

Practical reading

On average, c proposals are required per accepted sample. For a fixed q, smaller valid c improves efficiency.

This geometric-trial interpretation extends the efficiency discussion in Section 6.4.

Choosing the best envelope constant

The validity condition is

$$c \geq \sup_{x:q(x)>0} \frac{p(x)}{q(x)}.$$

For a fixed proposal q, the smallest possible envelope constant is

$$\boxed{c_{\min} = \sup_{x:q(x)>0} \frac{p(x)}{q(x)}.}$$

- Choosing $c < c_{\min}$ invalidates the algorithm.
- Choosing $c > c_{\min}$ is valid but wastes proposals.
- A good q makes $c_{\min}$ small while remaining easy to sample.

Important

The mathematical minimum is optimal *for this fixed proposal* q; finding a different q can improve the method further.

How should we choose the proposal q?

A useful proposal

- Is simple to sample from.
- Covers the whole target support.
- Roughly matches the target shape.
- Has tails sufficient for a finite bound c.

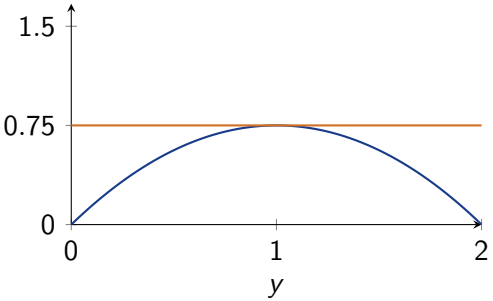
Failure mode

If $q(y) = 0$ for a region where $p(y) > 0$, the missing target values can never be proposed. No finite c can satisfy $p \leq cq$.

The notes focus on the univariate case: rejection sampling can become very inefficient as the dimension increases.

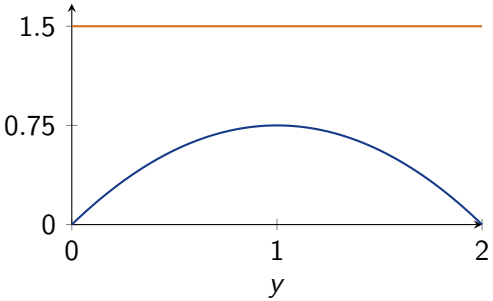
Comparing envelopes: smaller c is better

$c = 1.5$ **(tight)**



$\mathbb{P}(A) = 2/3$

$c = 3$ **(loose)**



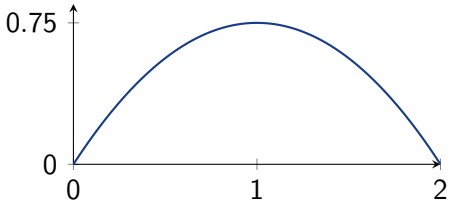
$\mathbb{P}(A) = 1/3$

Both plots use $q = \text{Unif}(0, 2)$ and the same target developed next.

Consider the target density

$$p(y) = \frac{3}{4} y(2-y)\mathbf{1}_{[0,2]}(y).$$

- Support: $0 \leq y \leq 2$.
- Concave quadratic, zero at $y = 0$ and $y = 2$.
- Symmetric about $y = 1$.



Check the target and find its maximum

First check that p integrates to one:

$$\int_0^2 \frac{3}{4} y(2-y)\, dy = \frac{3}{4}\left[y^2 - \frac{y^3}{3}\right]_0^2 = 1.$$

Differentiate on $[0, 2]$:

$$p'(y) = \frac{3}{4}(2-2y) = \frac{3}{2}(1-y).$$

So the maximum occurs at $y = 1$:

$$\boxed{\max_{y\in[0,2]} p(y) = p(1) = \frac{3}{4}.}$$

Next step

Choose a convenient proposal whose support contains $[0, 2]$.

Choose a uniform proposal and find c

A natural choice is $Y \sim \text{Unif}(0, 2)$, so

$$\boxed{q(y) = \frac{1}{2}\mathbf{1}_{[0,2]}(y).}$$

For $y \in (0, 2)$,

$$\frac{p(y)}{q(y)} = \frac{\frac{3}{4}y(2-y)}{\frac{1}{2}} = \frac{3}{2}y(2-y).$$

Since $\max_{[0,2]} y(2-y) = 1$,

$$\boxed{c_{\min} = \sup_y \frac{p(y)}{q(y)} = \frac{3}{2}.}$$

Then $cq(y) = \frac{3}{4}$ on the support: a rectangular envelope.

Acceptance rule becomes very simple

With

$$p(y) = \frac{3}{4}y(2-y), \quad q(y) = \frac{1}{2}, \quad c = \frac{3}{2},$$

we have

$$\begin{aligned}\alpha(y) &= \frac{p(y)}{cq(y)} \\ &= \frac{\frac{3}{4}y(2-y)}{\frac{3}{2}\cdot\frac{1}{2}} \\ &= \boxed{y(2-y)}.\end{aligned}$$

The algorithm for this example

Draw $Y \sim \text{Unif}(0,2)$ and $U \sim \text{Unif}(0,1)$.
Accept Y if and only if $U \leq Y(2-Y)$.

What is the acceptance rate in the example?

Because the envelope is valid with $c = 3/2$,

$$\boxed{\mathbb{P}(A) = \frac{1}{c} = \frac{2}{3} \approx 0.667.}$$

And the expected number of proposals per accepted sample is

$$\boxed{\mathbb{E}[T] = c = \frac{3}{2}.}$$

Geometric interpretation

The area under the target is 1. The rectangle $cq(y) = 3/4$ over $[0, 2]$ has area $2 \times 3/4 = 3/2$.

Acceptance rate = target area / envelope area $= 1/(3/2) = 2/3$.

Numerical accept–reject decisions

Use the rule $U \leq Y(2 - Y)$.

Candidate Y	Uniform U	$Y(2 - Y)$	Decision
0.20	0.50	0.36	Reject
0.70	0.80	0.91	Accept
1.50	0.55	0.75	Accept
1.80	0.40	0.36	Reject

Remember

Rejected candidates do not enter the final sample. Continue proposing until the desired number of *accepted* values is obtained.

Checking the samples against the target

For the quadratic target $p(y) = \frac{3}{4}y(2-y)$ on $[0,2]$,

$$\boxed{\mathbb{E}_p[Y] = 1, \qquad \mathrm{Var}_p(Y) = \frac{1}{5}.}$$

- Plot an empirical histogram of accepted values and superimpose $p(y)$.
- Check that the sample mean is near 1 and variance near 0.2.
- Compare observed acceptance fraction to $2/3$.

Key distinction

These empirical checks help detect implementation errors; they do *not* replace the mathematical proof in (6.5)–(6.7).

Common pitfalls

1. Wrong support: sampling $Y \sim \text{Unif}(0,1)$ cannot produce target values in $(1,2)$.
2. Invalid envelope: choosing $c < \sup p/q$ can make $p(y)/(cq(y)) > 1$.
3. Reusing uniforms: the acceptance uniform should be independent of the proposal.
4. Keeping rejected proposals: only accepted values follow p.
5. Inefficient envelope: an overly large c gives many more rejections.

Before running code

Always verify $p \leq cq$ on the *entire* target support.

Why focus on the univariate case?

- The proof works in higher dimensions as well, provided a valid envelope exists.
- In high dimensions, finding a good proposal q with a small c can be difficult.
- When c is large, the acceptance rate $1/c$ becomes very small.

Limitation from Section 6.4

Rejection sampling is often *inefficient for high-dimensional problems*; therefore, the lecture notes restrict attention to the univariate setting.

This motivates more advanced sampling methods studied later, including Markov chain Monte Carlo.

Quick exercise 1: choosing c

Consider the target and proposal

$$p(y) = 2y\,\mathbf{1}_{[0,1]}(y), \qquad q(y) = \mathbf{1}_{[0,1]}(y).$$

1. Verify that p is a density.
2. Find $c_{\min} = \sup p(y)/q(y)$.
3. Write down the acceptance test in terms of Y and U.
4. What is the theoretical acceptance rate?

Think: A proposal can have the correct support yet still require a nontrivial envelope constant.

$$\int_0^1 2y\, dy = [y^2]_0^1 = 1.$$

With $q(y) = 1$ on $[0, 1]$,

$$\boxed{c_{\min} = \sup_{0 \le y \le 1} 2y = 2.}$$

The acceptance probability at a candidate Y is

$$\frac{p(Y)}{cq(Y)} = \frac{2Y}{2} = Y.$$

Sampling rule

Draw $Y, U \overset{\text{iid}}{\sim} \text{Unif}(0, 1)$. Accept Y if $U \le Y$.

The overall acceptance rate is $\boxed{1/c = 1/2}$.

Summary: Section 6.4 rejection sampling

Ingredients

Target p; an easy-to-sample proposal q; c such that $p(x) \leq cq(x)$.

$$Y \sim q, \quad U \sim \text{Unif}(0,1), \quad \text{accept if } U \leq \frac{p(Y)}{cq(Y)}.$$

- **Correctness:** Accepted samples have target density p (equations 6.5–6.7).
- **Efficiency:** Acceptance probability is exactly $1/c$.
- **Design rule:** Choose a good proposal and the smallest valid c.
- **Limitation:** Efficiency can deteriorate badly in higher dimensions.

Next topic: Section 7 — Markov Chain Monte Carlo.