Recall that our Monte Carlo objective is often

$$\mathcal{I} = \mathbb{E}_p[f(X)] = \int f(x)p(x)\,dx.$$

To form a Monte Carlo estimator, we need samples $X_1, \ldots, X_N$ from p.

Section 6.2: Box–Muller

Transforms two independent uniforms into two independent *normal* random variables.

Section 6.3: inverse transform

Uses the CDF of a target *univariate* distribution to transform one uniform random variable.

Question: Can we transform $U \sim \text{Unif}(0, 1)$ into $X \sim F$?

Recall: the cumulative distribution function

For a real-valued random variable X, its **CDF** is

$$\boxed{F(x) = \Pr(X \leq x).}$$

- F is non-decreasing, with values in $[0, 1]$.
- $F(x)$ measures the probability accumulated to the *left* of x.
- If X has density p, then

$$F(x) = \int_{-\infty}^{x} p(t)\, dt, \qquad F'(x) = p(x)$$

where the derivative exists.

Sampling problem

Given the target CDF F, how can a uniform number $u \in (0, 1)$ be converted into a value x with that CDF?

The inverse CDF (quantile function)

Assumption used in Section 6.3 of the notes

Work with a target distribution supported on $D \subset \mathbb{R}$ whose continuous CDF F is strictly increasing and therefore invertible on its range.

For $0 < u < 1$, define the **quantile**

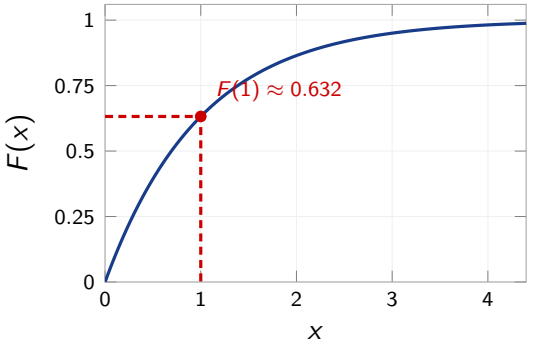
$$\boxed{x = F^{-1}(u).}$$

This is the value x at which the accumulated probability reaches u:

$$\underbrace{F(x)}_{\text{cumulative probability}} = u.$$

- Input: a probability level u.
- Output: the corresponding value x of the target variable.

Graphical intuition: reading a quantile from the CDF



1. Pick u on the vertical probability axis.
2. Move *horizontally* to the CDF curve.
3. Move *vertically* down to the x-axis.

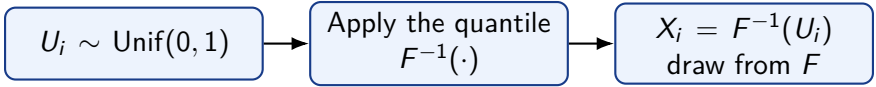
Read-off

$u = F(x)$, so

$$\boxed{x = F^{-1}(u)}.$$

Inverse-CDF method (as in the lecture notes)

1. Generate independent $U_1, \ldots, U_N \sim \text{Unif}(0,1)$.
2. For each $i = 1, \ldots, N$, compute $X_i = F^{-1}(U_i)$.
3. Use $X_1, \ldots, X_N$ as samples from the target distribution F.



One uniform draw produces one target-distribution draw.

Let

$$U \sim \text{Unif}(0,1), \qquad X = F^{-1}(U).$$

To show that X has target CDF F, fix an arbitrary value a.

$$\begin{aligned} \Pr(X \le a) &= \Pr(F^{-1}(U) \le a) \\ &= \Pr(U \le F(a)) \qquad \text{(because } F \text{ is increasing).} \end{aligned}$$

Key equivalence

$$\boxed{F^{-1}(U) \le a \iff U \le F(a).}$$

The second event is easy to evaluate because U is uniform.

Proof: the sampled variable has CDF F

Continue the argument from the previous slide:

$$\begin{aligned}\Pr(X \leq a) &= \Pr(F^{-1}(U) \leq a) \\ &= \Pr(U \leq F(a)) \\ &= F(a).\end{aligned}$$

The last equality follows because, for $0 \leq v \leq 1$,

$$U \sim \text{Unif}(0,1) \quad \Longrightarrow \quad \Pr(U \leq v) = v.$$

Conclusion

$$\boxed{X = F^{-1}(U) \text{ has CDF } F.}$$

A second view: the density transformation

Additional teaching example

Suppose the CDF is differentiable and strictly increasing, with density $p(x) = F'(x)$. For the monotone transformation $X = F^{-1}(U)$,

$$u = F(x), \qquad \left|\frac{du}{dx}\right| = F'(x) = p(x).$$

By the one-dimensional change-of-variables formula,

$$\begin{aligned} p_X(x) &= p_U(F(x)) \left|\frac{dF(x)}{dx}\right| \\ &= 1 \cdot p(x) = p(x). \end{aligned}$$

Interpretation

The quantile transformation stretches or compresses intervals so that a uniform probability mass becomes the desired target density.

Worked example 1: uniform distribution on $[a, b]$

Additional teaching example

Suppose $X \sim \text{Unif}(a, b)$, with $a < b$. Its CDF is

$$F(x) = \begin{cases} 0, & x < a, \\ \dfrac{x-a}{b-a}, & a \le x \le b, \\ 1, & x > b. \end{cases}$$

For $0 < u < 1$, solve $u = (x - a)/(b - a)$:

$$\boxed{F^{-1}(u) = a + (b - a)u.}$$

Sampler

Draw $U \sim \text{Unif}(0, 1)$, then return $\boxed{X = a + (b - a)U}$.

Example 1: evaluate the quantile numerically

Suppose the target distribution is Unif$(2, 6)$. Then

$$X = 2 + 4U.$$

Uniform input u	Calculation	Output x
0.10	$2 + 4(0.10)$	2.40
0.25	$2 + 4(0.25)$	3.00
0.80	$2 + 4(0.80)$	5.20

Quick check

All outputs belong to $[2, 6]$. Moreover, the mapping is increasing: larger uniforms give larger target values.

Worked example 2: exponential distribution

Additional teaching example

Let $X \sim \text{Exp}(\lambda)$, where $\lambda > 0$, with CDF

$$F(x) = 1 - e^{-\lambda x}, \qquad x \geq 0.$$

Set $u = F(x)$ and solve for x:

$$\begin{aligned} u &= 1 - e^{-\lambda x} \\ e^{-\lambda x} &= 1 - u \\ -\lambda x &= \log(1-u) \\ x &= -\frac{1}{\lambda}\log(1-u). \end{aligned}$$

Inverse-transform sampler

$$\boxed{X = -\frac{1}{\lambda}\log(1-U), \qquad U \sim \text{Unif}(0,1).}$$

Exponential example: a numerical draw

Suppose $X \sim \text{Exp}(2)$ and our uniform draw is $u = 0.75$. Then

$$\begin{aligned} x &= F^{-1}(0.75) \\ &= -\frac{1}{2}\log(1 - 0.75) \\ &= -\frac{1}{2}\log(0.25) \\ &= \log 2 \approx \boxed{0.6931}. \end{aligned}$$

Check using the CDF

$$F(\log 2) = 1 - e^{-2\log 2} = 1 - \frac{1}{4} = 0.75.$$

The CDF check is a useful way to verify an inverse formula.

Why do some exponential formulas use $-\log U$?

If $U \sim \text{Unif}(0,1)$, then $1 - U \sim \text{Unif}(0,1)$ as well. Therefore,

$$-\frac{1}{\lambda}\log(1-U) \quad \stackrel{d}{=} \quad -\frac{1}{\lambda}\log U.$$

- The two expressions have the **same distribution**.
- They generally produce *different numerical outputs* for the same input u.
- Numerical code must avoid $\log 0$; library methods such as `log1p(-u)` can improve accuracy when u is small.

Not the same as equality for every draw

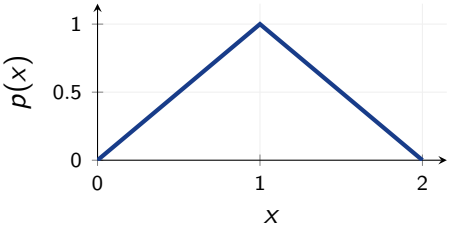
Equality in distribution, $\stackrel{d}{=}$, does not mean that the two transformed values are identical when the same u is used.

Worked example 3: a triangular distribution

Additional teaching example

Consider a distribution with density

$$p(x) = \begin{cases} x, & 0 \le x \le 1, \\ 2 - x, & 1 < x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$



Integrate the density to obtain the CDF:

$$F(x) = \begin{cases} \frac{x^2}{2}, & 0 \le x \le 1, \\ 1 - \frac{(2-x)^2}{2}, & 1 < x \le 2. \end{cases}$$

Triangular example: invert the CDF piecewise

For $0 < u \leq \frac{1}{2}$, the corresponding x lies in $[0, 1]$:

$$u = \frac{x^2}{2} \quad \Longrightarrow \quad \boxed{x = \sqrt{2u}}.$$

For $\frac{1}{2} < u < 1$, the corresponding x lies in $(1, 2]$:

$$u = 1 - \frac{(2-x)^2}{2} \quad \Longrightarrow \quad \boxed{x = 2 - \sqrt{2(1-u)}}.$$

Complete quantile function

$$\boxed{F^{-1}(u) = \begin{cases} \sqrt{2u}, & 0 < u \leq \frac{1}{2}, \\ 2 - \sqrt{2(1-u)}, & \frac{1}{2} < u < 1. \end{cases}}$$

Triangular example: choose the correct branch

Take two uniform inputs:

Input $u = 0.18$

Since $0.18 < \frac{1}{2}$,

$$\begin{aligned} x &= \sqrt{2(0.18)} \\ &= \sqrt{0.36} \\ &= \boxed{0.60}. \end{aligned}$$

Input $u = 0.875$

Since $0.875 > \frac{1}{2}$,

$$\begin{aligned} x &= 2 - \sqrt{2(1 - 0.875)} \\ &= 2 - \sqrt{0.25} \\ &= \boxed{1.50}. \end{aligned}$$

Common error

Do not apply a single branch of a piecewise inverse CDF to every uniform input.

What if the inverse CDF is not elementary?

The notes point out a key practical limitation: inverting an arbitrary CDF can be expensive.

- Some CDFs have simple inverses: uniform, exponential, certain power laws.
- Others have no elementary closed-form inverse. For example, the standard normal CDF

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$$

  requires specialised evaluation to compute $\Phi^{-1}(u)$.
- A numerical inversion may require solving

$$\boxed{F(x) - u = 0}$$

  for each individual draw.

Cost consideration

A conceptually simple formula, $X = F^{-1}(U)$, can still be expensive to implement.