

Problem Set 2

Question 1. Show that the following distributions belong to a natural exponential family (NEF), with suitable reparametrization. *Hint: note that reparametrization means that the current parameters may not be suitable for showing the NEF form; you may need to redefine distribution parameters. For example, in 1.(a), we need to do the following reparametrization $\theta = \log(\lambda)$.*

(a) The Poisson distribution:

$$p(x, \lambda) = \frac{\lambda^x \exp(-\lambda)}{x!}$$

(b) The Gamma distribution with α fixed:

$$f(x, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x > 0$$

where:

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt \text{ is the Gamma function.}$$

(c) The Inverse Gaussian distribution with λ fixed:

$$f(x|\mu, \lambda) = \left(\frac{\lambda}{2\pi x^3} \right)^{1/2} \exp \left(-\frac{\lambda(x - \mu)^2}{2\mu^2 x} \right), \quad x > 0$$

Question 2. Show that the univariate normal distribution with both mean and variance being varying parameters, are in an exponential family.

Solutions

1.(a) Poisson Distribution

The probability mass function (PMF) of the Poisson distribution is given by:

$$p(x, \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \dots$$

Taking the logarithm of the PMF:

$$\log p(x, \lambda) = x \log(\lambda) - \lambda - \log(x!)$$

Thus, the Poisson distribution can be written in the following exponential family form:

$$p(x, \lambda) = \frac{1}{x!} \exp(x \log(\lambda) - \lambda)$$

Now we let $\theta = \log(\lambda)$, we obtain the natural exponential family form as:

$$p(x, \theta) = \frac{1}{x!} \exp(x\theta - e^\theta)$$

where $\theta = \log(\lambda)$ is the distribution parameter.

1.(b) Gamma Distribution

The probability density function (PDF) of the Gamma distribution with shape parameter α and rate parameter β is given by:

$$f(x|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x > 0$$

Taking the logarithm of the PDF:

$$\log f(x|\alpha, \beta) = (\alpha - 1) \log(x) - \beta x + \alpha \log(\beta) - \log(\Gamma(\alpha))$$

Rewriting this expression in the form of the natural exponential family:

$$f(x|\alpha, \beta) = \exp((\alpha - 1) \log(x) - \beta x + \alpha \log(\beta) - \log(\Gamma(\alpha)))$$

In natural exponential family form, this can be written as:

$$f(x|\theta) = \exp(\theta x - b(\theta)) h(x)$$

where:

$$\begin{aligned} \theta &= -\beta, \\ b(\theta) &= \alpha \log(-\theta), \\ h(x) &= \frac{x^{\alpha-1}}{\Gamma(\alpha)}. \end{aligned}$$

1.(c) Inverse Gaussian Distribution

The probability density function (PDF) of the Inverse Gaussian distribution is given by:

$$f(x|\mu, \lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right), \quad x > 0$$

This can be rewritten as:

$$\begin{aligned} f(x|\mu, \lambda) &= \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left(-\frac{\lambda}{2}\left(\frac{x}{\mu^2} - \frac{2}{\mu} + \frac{1}{x}\right)\right) \\ &= \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left(-\frac{\lambda}{2x}\right) \exp\left(-\frac{\lambda}{2}\left(\frac{x}{\mu^2} - \frac{2}{\mu}\right)\right) \\ &= \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left(-\frac{\lambda}{2x}\right) \exp\left(-\frac{\lambda}{2\mu^2}x + \frac{\lambda}{\mu}\right) \end{aligned}$$

We can now express the Inverse Gaussian PDF in the natural exponential family form:

$$f(x|\theta, \lambda) = h(x) \exp(\theta x - A(\theta))$$

where:

- $\theta = -\frac{\lambda}{2\mu^2},$
- $A(\theta) = \sqrt{-2\theta\lambda},$
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$$h(x) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left(-\frac{\lambda}{2x}\right)$$

2.

The probability density function (PDF) of the univariate Gaussian (normal) distribution with mean μ and variance σ^2 is given by:

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

We will demonstrate that this distribution belongs to the general exponential family, but not the natural exponential family (NEF) when both μ and σ^2 are varying parameters.

We can rewrite the PDF in exponential family form as follows:

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Expanding the quadratic term:

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2} + \frac{\mu x}{\sigma^2} - \frac{\mu^2}{2\sigma^2}\right)$$

This can be written in exponential family form:

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{\mu x}{\sigma^2} - \frac{\mu^2}{2\sigma^2} - \log(\sigma) - \frac{x^2}{2\sigma^2}\right)$$

Identifying the components of the exponential family:

$$f(x|\mu, \sigma^2) = h(x) \exp(\eta_1(\mu, \sigma^2)T_1(x) + \eta_2(\sigma^2)T_2(x) - A(\mu, \sigma^2))$$

where:

$$\eta_1(\mu, \sigma) = \frac{\mu}{\sigma^2} \quad (\text{natural parameter for mean}),$$

$$\eta_2(\mu, \sigma) = -\frac{1}{2\sigma^2} \quad (\text{natural parameter for variance}),$$

$$T_1(x) = x \quad (\text{sufficient statistic for mean}),$$

$$T_2(x) = x^2 \quad (\text{sufficient statistic for variance}),$$

$$h(x) = \frac{1}{\sqrt{2\pi}} \quad (\text{base measure}),$$

$$A(\mu, \sigma) = \frac{\mu^2}{2\sigma^2} + \log(\sigma).$$

Note that in the equation above, we can take $\theta_1 = \mu$ and $\theta_2 = \sigma$. $\eta(\mu, \sigma)$ is a vector-valued function whose output is two dimensional (η_1 and η_2). $T(x)$ is also a vector-valued function whose output is two dimensional ($T_1(x)$ and $T_2(x)$). This shows that the normal distribution with varying μ and σ^2 is in an exponential family.