Solutions for Chapter 5 – Week 8

1. The posterior distribution is given by $\pi(p \mid \boldsymbol{y}) \propto \pi(\boldsymbol{y} \mid p)\pi(p)$. The prior density is proportional to
$$\pi(p) \propto p^{\alpha-1}(1-p)^{\beta-1}.$$
Therefore, the posterior density is proportional to
$$\begin{aligned}\pi(p \mid \boldsymbol{y}) &\propto \pi(p)\prod_{i=1}^{N}\pi(y_i \mid p)\\ &\propto (1-p)^{\sum y_i - N}p^N p^{\alpha-1}(1-p)^{\beta-1}\\ &= p^{N+\alpha-1}(1-p)^{\sum y_i - N+\beta-1},\end{aligned}$$
which we recognise as the density of a Beta$(N+a, \sum_{i=1}(y_i-1)+\beta)$ distribution. To propose new values for p we use a uniform proposal distribution $p \sim U[p-\varepsilon, p-\varepsilon]$ for some moderate $\varepsilon > 0$. Since this is symmetric around p, the proposal density ratio cancels out and the acceptance probability for $p' \in (0,1)$ is given by
$$\begin{aligned}p_{\text{acc}} &= \min\left\{1, \frac{\pi(p' \mid \boldsymbol{y})}{\pi(p \mid \boldsymbol{y})}\right\}\\ &= \min\left\{1, \frac{p'^{N+\alpha-1}(1-p')^{\sum y_i - N+\beta-1}}{p^{N+\alpha-1}(1-p)^{\sum y_i - N+\beta-1}}\right\}\\ &= \min\left\{1, \left(\frac{p'}{p}\right)^{N+\alpha-1}\left(\frac{1-p'}{1-p}\right)^{\sum y_i - N+\beta-1}\right\}\end{aligned}$$
and 0 otherwise.

An MCMC algorithm could look like this:

1. Set $p^{(0)} = 0.5$ and $i = 1$
2. For $p = p^{(i-1)}$ propose a new value $p' \sim U[p-\varepsilon, p+\varepsilon]$. If $p' \notin (0,1)$ reject straight away and go to step 4.
3. Accept p' with probability p_{acc}
4. If accepted, set $p^{(i)} = p'$ otherwise $p^{(i)} = p^{(i-1)}$
5. Repeat steps 2 -4 for $i = 2, \ldots, n$.

2. By Bayes' theorem, the posterior distribution is proportional to $\pi(\beta \mid \boldsymbol{y}) \propto \pi(\boldsymbol{y} \mid \beta)\pi(\beta)$. The likelihood function is given by
$$\pi(\boldsymbol{y} \mid \beta) = \prod_{i=1}^{N}\frac{\beta}{y_i^{\beta+1}} = \frac{\beta^N}{\prod_{i=1}^{N} y_i^{\beta+1}}.$$

The prior density is proportional to $\pi(\beta) \propto \beta^{a-1}\mathrm{e}^{-b\beta}$. The posterior density is therefore proportional to

$$\pi(\beta \mid \boldsymbol{y}) \propto \frac{\beta^{N+a-1}e^{-b\beta}}{\prod y_i^{\beta+1}}.$$

If we use a Metropolis-Hastings Random walk algorithm and propose new values for β by $\beta' \sim N(\beta, \sigma^2)$, the acceptance probability is

$$\begin{aligned} p_{\text{acc}} &= \min\left\{1, \frac{\pi(\beta' \mid \boldsymbol{y})}{\pi(\beta \mid \boldsymbol{y})}\right\} \\ &= \min\left\{1, \frac{\beta'^{N+a-1}e^{-b\beta'}}{\prod y_i^{\beta'+1}} \frac{\prod_{i=1}^N y_i^{\beta+1}}{\beta^{N+a-1}e^{-b\beta}}\right\} \\ &= \min\left\{1, \left(\frac{\beta'}{\beta}\right)^{N+\alpha-1} e^{-b(\beta'-\beta)} \prod_{i=1}^N (y_i)^{\beta-\beta'}\right\}. \end{aligned}$$

for $\beta' > 0$ and 0 otherwise.

An MCMC algorithm could look like this:

1. Set $\beta^{(0)} = 2$ and $i = 1$
2. For $\beta = \beta^{(i-1)}$ propose a new value β' by $\beta' \sim N(\beta, \sigma^2)$
3. Accept β' with probability p_{acc}
4. If accepted, set $\beta^{(i)} = \beta'$ otherwise $\beta^{(i)} = \beta^{(i-1)}$
5. Repeat steps 2–4 for $i = 1, \ldots, n$.