Solutions for Chapter 5 – Week 9

1. (a) By Bayes' theorem, the posterior distribution is $\pi(\alpha, \beta \mid \boldsymbol{y}) \propto \pi(\boldsymbol{y} \mid \alpha, \beta)\pi(\alpha)\pi(\beta)$. The likelihood function for $\alpha, \beta > 0$ is

$$\pi(\boldsymbol{y} \mid \beta, \alpha) = \prod_{i=1}^{N} \frac{\beta \alpha^{\beta}}{y_i^{\beta+1}} = \frac{\beta^N \alpha^{N\beta}}{\prod_{i=1}^{N} y_i^{\beta+1}}.$$

The prior distributions $\pi(\alpha) \propto \exp(-c\alpha)$ and $\pi(\beta) \propto \beta^{a-1} \exp(-b\beta)$. Thus the posterior distribution is

$$\pi(\alpha, \beta \mid \boldsymbol{y}) \propto \frac{\beta^N \alpha^{N\beta}}{\prod y_i^{\beta+1}} \exp(-c\alpha)\beta^{a-1} \exp(-b\beta).$$

Considering only terms that depend on α gives the full conditional distribution:

$$\pi(\alpha \mid \beta, \boldsymbol{y}) \propto \alpha^{N\beta} \exp(-c\alpha)$$

This is the form of a Gamma distribution, i.e., $\alpha \mid (\beta, \boldsymbol{y}) \sim \Gamma(N\beta + 1, c)$. Considering only terms that depend on β gives the full conditional distribution

$$\pi(\beta \mid \alpha, \boldsymbol{y}) \propto \frac{\beta^{N+a-1} \alpha^{N\beta}}{\prod y_i^{\beta+1}} \exp(-b\beta).$$

This distribution has no closed form.

(b) To generate samples from the posterior distribution, we can use a Metropolis-Hastings Random Walk algorithm to generate samples for β and a Gibbs sampler to generate samples for α. We can adapt the acceptance probability from (a) to get the acceptance probability for this MCMC algorithm

$$p_{\text{acc}}(\alpha, \beta, \beta') = \min\left\{1, \left(\frac{\beta'}{\beta}\right)^{N+a-1} \alpha^{N(\beta'-\beta)} e^{-b(\beta'-\beta)} \prod (y_i)^{\beta-\beta'}\right\}$$

if $\beta' > 0$ and 0 otherwise.

An example MCMC algorithm is

1. Set $\beta^{(0)} = 2$, $\alpha^{(0)} = 2$ and $i = 1$
2. Propose a new value by $\beta' \sim N(\beta^{(i-1)}, \sigma^2)$
3. Accept β' with probability $p_{\text{acc}}(\alpha^{(i-1)}, \beta^{(i-1)}, \beta')$
4. If accepted, set $\beta^{(i)} = \beta'$ otherwise $\beta^{(i)} = \beta^{(i-1)}$
5. Sample a value for $\alpha^{(i)}$ by drawing from a $\Gamma(N\beta^{(i)} + 1, c)$ distribution.
6. Repeat steps 2 - 5 for $i = 2, \ldots, n$.

(c) To check the mixing of this algorithm, we can look at trace plots to determine if our choice of σ^2 is adequate. We can also use the trace plot to check if a longer burn-in period might be necessary.

2. (a) An example MCMC algorithm is
   1. Set $\theta_1^{(0)} = \alpha$, $\theta_2^{(0)} = \beta$ and $i = 1$
   2. Sample a value for $\theta_1^{(i)}$ by drawing the distribution $\theta_1 \mid \theta_1^{(i-1)}, \boldsymbol{y}$.
   3. Sample a value for $\theta_2^{(i)}$ by drawing the distribution $\theta_2 \mid \theta_1^{(i)}, \boldsymbol{y}$.
   4. Repeat steps 2 - 3 for $i = 2, \ldots, n$.

   (b) We are proposing the value θ_1' from the distribution $\theta_1 \mid \theta_2, \boldsymbol{y}$. The acceptance probability is given by

$$\begin{aligned} p_{\text{acc}} &= \min\left\{1, \frac{\pi(\theta_1', \theta_2 \mid \boldsymbol{y})}{\pi(\theta_1, \theta_2 \mid \boldsymbol{y})} \frac{q(\theta_1 \mid \theta_1')}{q(\theta_1' \mid \theta_1)}\right\} \\ &= \min\left\{1, \frac{\pi(\theta_1' \mid \theta_2, \boldsymbol{y})}{\pi(\theta_1 \mid \theta_2, \boldsymbol{y})} \frac{\pi(\theta_1 \mid \theta_2, \boldsymbol{y})}{\pi(\theta_1' \mid \theta_2, \boldsymbol{y})}\right\} \\ &= 1, \end{aligned}$$

   where we used that

$$\frac{\pi(\theta_1', \theta_2 \mid \boldsymbol{y})}{\pi(\theta_1, \theta_2 \mid \boldsymbol{y})} = \frac{\pi(\theta_1' \mid \theta_2, \boldsymbol{y})\pi(\theta_2 \mid \boldsymbol{y})}{\pi(\theta_1 \mid \theta_2, \boldsymbol{y})\pi(\theta_2 \mid \boldsymbol{y})} = \frac{\pi(\theta_1' \mid \theta_2, \boldsymbol{y})}{\pi(\theta_1 \mid \theta_2, \boldsymbol{y})}.$$