Problem Sheet for Chapter 5 – Week 8

1. Suppose that $Y_1, \dots, Y_N \mid p \overset{\text{iid}}{\sim} \text{Geom}(p)$ for $p \in (0,1)$. The likelihood of a data sample is therfore given by

$$\pi(y \mid p) = \begin{cases} (1-p)^{y-1}p, & y \in \mathbb{N}, \\ 0, & \text{else.} \end{cases}$$

   Use a $\text{Beta}(\alpha, \beta)$ prior distribution to obtain the posterior distribution. Although this distribution has a closed form, develop a Metropolis–Hastings Random Walk algorithm with an appropriate symmetric proposal distribution to generate samples from the posterior distribution.

2. The density function for the Pareto distribution $\text{Pareto}(\beta)$ with scale $\alpha = 1$ and shape $\beta > 0$ is given by

$$\pi(y \mid \beta) = \frac{\beta}{y^{\beta+1}} \mathbf{1}_{(1,\infty)}(y), \quad y \in \mathbb{R}$$

   Suppose that $Y_1, \dots, Y_n \mid \beta \overset{\text{iid}}{\sim} \text{Pareto}(\beta)$ and denote the observed data by $\boldsymbol{y} = \{y_1, \dots, y_N\}$. Place a Gamma prior distribution on β such that $\beta \sim \Gamma(a, b)$.

   (a) Show that the posterior distribution has no closed form and is proportional to

$$\pi(\beta \mid \boldsymbol{y}) \propto \frac{\beta^{N+a-1} e^{-b\beta}}{\prod y_i^{\beta+1}}.$$

   (b) Construct a Metropolis–Hastings Random walk algorithm with proposal distribution $q(\beta' \mid \beta) \sim N(\beta, \sigma^2)$ for some $\sigma^2 > 0$, to generate samples from this distribution, making sure to derive the acceptance probability.