Problem Sheet for Week 4 & 5

1. Suppose that $Y_1, \dots, Y_N \mid p \overset{i.i.d.}{\sim} \text{Binom}(n, p)$.
   (a) By placing a Beta prior distribution on p such that $p \sim \text{Beta}(\alpha, \beta)$ derive the posterior distribution.
   (b) Suppose that y_{N+1} is then observed and is also independently drawn from the same distribution. Derive the posterior distribution $\pi(p \mid y_1, \dots, y_N, y_{N+1})$ by updating the posterior distribution in part (a).
   (c) Show that you obtain the same distribution if you observe all $N + 1$ data points at the start of the process.

2. Let $Y_1, \dots, Y_N \mid \lambda \overset{i.i.d.}{\sim} \text{Poisson}(\lambda)$.
   (a) Suppose we place a Gamma prior distribution on λ, where $\lambda \sim \text{Gamma}(\alpha, \beta)$. Derive the posterior distribution.
   (b) Fix $\alpha = 1$. Discuss the effects of β on the posterior distribution.
   (c) Without fixing α, derive the posterior predictive distribution for a new observation $\tilde{y}$. *Hint: the Negative Binomial distribution with parameters r and p has probability mass function*

$$\pi(k \mid r, p) = \frac{\Gamma(k + r)}{\Gamma(r) k!} (1 - p)^k p^r,$$

   for positive integers k.

3. Let $X \sim \pi(x)$ be a continuous random variable and $Y = h(X)$ for some strictly monotonic and smooth function h. Show that Y has density

$$\pi(y) = \pi(x) \left| \frac{\partial x}{\partial y} \right|.$$

   Now, suppose that $X \sim \text{Exp}(1)$. Find the density function $\pi(y)$ for $Y = \sqrt{X}$.

4. Let $X_1 \dots, X_n \mid \lambda \overset{i.i.d.}{\sim} \text{Exp}(\lambda)$
   (a) Construct an invariant prior distribution for λ.
   (b) Derive the posterior distribution using the invariant prior distribution.
   (c) What can you notice about this prior distribution?