Solutions for Chapter 4 – Week 3

1. This exercise is about the inverse transform sampling method.

   (a) Using the fact that necessarily $\int_0^4 \pi(y)dy = 1$, we have $\alpha = \frac{1}{64}$.

   (b) For $y \in [0, 4]$, we have

   $$\int_{-\infty}^{y} \pi(x)\, dx = \int_0^y \frac{1}{64} x^3 dx = \frac{1}{256} y^4.$$

   Hence the distribution function for Y is given by

   $$F(y) = \begin{cases} 0 & \text{for } y < 0 \\ \frac{1}{256} y^4 & \text{for } 0 \le y \le 4 \\ 1 & y > 4 \end{cases}.$$

   (c) For $x \in (0, 1]$ we have

   $$F(y) = \frac{1}{256} y^4 = x \iff y = 4x^{\frac{1}{4}},$$

   which shows that

   $$F^{-1}(x) = \begin{cases} -\infty & \text{for } x = 0 \\ 4x^{\frac{1}{4}} & \text{for } 0 < x \le 1 \end{cases}.$$

   By the inverse transform theorem, if $U \sim U[0, 1]$ then $4U^{\frac{1}{4}}$ has the same distribution as Y.

2. Another exercise about the inverse transform method, but this time with a slightly more complex density function.

   (a) For $y > 0$ we first need to evaluate

   $$F(y) = \int_0^y \frac{1}{2\sqrt{x}} e^{-\sqrt{x}}\, dx.$$

   The easiest substitution to use is $u = \sqrt{x}$. This gives us $dx = 2du\sqrt{x}$. Substituting these into the integral gives

   $$\begin{aligned} \int_0^y \frac{1}{2u} e^{-u}\, du &= \int_0^{\sqrt{y}} e^{-u}\, du \\ &= [-e^{-u}]_0^{\sqrt{y}} \\ &= 1 - e^{-\sqrt{y}}. \end{aligned}$$

The distribution function is therefore

$$F(y) = \begin{cases} 1 - e^{-\sqrt{y}} & \text{for } y > 0 \\ 0 & \text{otherwise} \end{cases}.$$

(b) Using the inverse transform method, if $U \sim U[0,1]$ then $F^{-1}(U)$ will have the same distribution as Y. The inverse of the distribution function on $(0,1)$ is

$$F^{-1}(y) = (\log(1-y))^2.$$

The method would be
i. Sample $u \sim U[0,1]$ ii. Compute $y = (\log(1-u))^2$

3. The normal distribution is difficult to sample from, especially as the inverse of its distribution is difficult to work with. One method to generate samples is using a rejection sampling method.

(a) The density function of a exponential distribution with rate λ is $q(x \mid \lambda) = \lambda e^{-\lambda x}$ for $x \geq 0$. The ratio is given by

$$\begin{aligned} \frac{\pi(x)}{q(x)} &= \frac{\frac{2}{\sqrt{2\pi}} \exp\{-\frac{1}{2}x^2\}}{\lambda e^{-\lambda x}} \\ &= \frac{2}{\lambda\sqrt{2\pi}} \exp\left\{\lambda x - \frac{1}{2}x^2\right\}. \end{aligned}$$

(b) To find the maximum value of this ratio, we first need to find the value of x that maximises the ratio. Differentiating gives

$$\frac{d}{dx} \frac{2}{\lambda\sqrt{2\pi}} \exp\left\{\lambda x - \frac{1}{2}x^2\right\} = \frac{2(\lambda - x)}{\lambda\sqrt{2\pi}} \exp\left\{\lambda x - \frac{1}{2}x^2\right\}$$

Setting this equal to 0 shows that $x = \lambda$ maximises the ratio. The maximum value of the ratio is thus

$$M = \frac{\pi(\lambda)}{q(\lambda)} = \frac{2}{\lambda\sqrt{2\pi}} \exp\left\{\frac{1}{2}\lambda^2\right\}$$

Since

(c) Since $\pi \leq Mq$ with the choice of M from part (a), a valid rejection sampling algorithm for the half normal distribution is given by:
i. Sample x from the exponential distribution.
ii. Sample $u \sim U[0,1]$.
iii. If $u \leq \frac{\pi(x)}{Mq(x)}$, then accept x as a sample from the half normal distribution.
Since M is the smallest value c s.t. $\pi \leq Mq$ and the average runtime is given by $1/c$, this rejection algorithm has minimal runtime.