Problem Sheet for Chapter 4 – Week 3

1. Let Y be a random variable with a probability density function defined by

$$\pi(y) = \alpha y^3 \mathbf{1}_{[0,4]}(y), \quad y \in \mathbb{R},$$

where $\alpha \in \mathbb{R}$ is a constant.

   (a) Compute the value of α.

   (b) Derive the distribution function $F(y)$.

   (c) Using the inverse transform theorem, derive a function g such that if $U \sim U[0,1]$, then $g(U)$ has the same distribution as Y.

2. Suppose Y has the density function

$$f(y) = \frac{1}{2\sqrt{y}} e^{-\sqrt{y}} \mathbf{1}_{(0,\infty)}(y), \quad y \in \mathbb{R}$$

   (a) Using integration by substitution, derive the distribution function $F(y)$.

   (b) Using the inverse transform method, construct a method for sampling from this distribution.

3. The density function for the half-normal distribution with variance 1 is

$$\pi(x) = \frac{2}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) \mathbf{1}_{[0,\infty)}(x), \quad x \in \mathbb{R}.$$

   (a) Using an exponential distribution with rate λ as a proposal distribution, show that

$$\frac{\pi(x)}{q(x)} = \frac{2}{\lambda\sqrt{2\pi}} \exp\left(\lambda x - \frac{1}{2}x^2\right).$$

   (b) What is M, the maximum value of this ratio? On the same plot, sketch $\pi(x)$ and $Mq(x)$.

   (c) Construct a rejection sampling algorithm with minimal average runtime to sample from the half normal distribution.