Problem Sheet for Chapter 3 - week 2

1. Suppose $Y_1, \dots, Y_N \mid p \overset{i.i.d}{\sim} \text{Geom}(p)$. Let $Y = (Y_1, \dots, Y_N)$ and we observe the data $y = (y_1, \dots, y_N)$. The density function for the Geometric distribution is

$$\pi(x \mid p) = (1-p)^{x-1}p, \qquad p \in (0,1],\, x \in \{1,2,3,4,\dots\}$$

   (a) Derive the likelihood function and then the maximum likelihood estimate $\hat{p}(y)$ for p. (You do not need to verify the stationary point is the maximum)

   (b) By letting $p \sim \text{Beta}(\alpha, \beta)$, derive the posterior distribution $\pi(p \mid y)$.

   (c) Compare the maximum likelihood estimate with with expectation of the posterior distribution. What values of α and β result in a posterior expectation that is equal to the maximum likelihood estimate? What happens to the prior distribution in this case?

2. When someone is infected with a disease, it's common to model their infectious period with a Gamma distribution. Suppose you observe data from 100 infected individuals with $\sum_{i=1}^{100} t_i$ = 870 days, where each t_i represents the number of days that they are infectious. Based on advice from clinicians, you model $T_1 \dots, T_n \mid \theta \overset{i.i.d}{\sim} \text{Gamma}(5, \theta)$. Using $\theta \sim \text{Exp}(0.01)$ as the prior distribution, derive the posterior distribution. Obtain the 95% credible interval for θ using R.

3. For each of the following statements, decide if they are true or false.

   (a) The likelihood function is proportional to the posterior distribution.

   (b) A 99% credible interval captures 99% of the posterior probability.

   (c) If a set of random variables are exchangeable, then we can reorder them without changing their joint distribution.

   (d) Bayesian and frequentist methods always lead to significantly different estimates.

4. The density function for the Pareto distribution with scale $\alpha = 1$ and shape β is given by

$$\pi(x \mid \alpha = 1,\, \beta) = \frac{\beta}{x^{\beta+1}}, \qquad x > 1, \beta > 0.$$

   Suppose the data, denoted by $\boldsymbol{y} = \{y_1, \dots, y_N\}$, are generated from a Pareto distribution with scale $\alpha = 1$ and shape β in an i.i.d manner. Place a Gamma prior distribution on β such that $\beta \sim \text{Gamma}(a, b)$. Derive the posterior distribution for β given the data, $\pi(\beta \mid \boldsymbol{y})$.